Technische Universität München
Zentrum Mathematik

Processes with reinforcement

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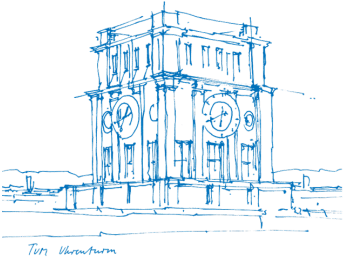
Firenze, March 22, 2019



Overview

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An undirected weighted graph

Let $G = (V, E)$ be a locally finite connected graph with vertex set V and set E of undirected edges.

You can think of

- your favorite graph,
- a finite box in $\mathbb{Z}^d$, or
- the integer lattice $\mathbb{Z}^d$.

Every edge $e \in E$ is given a weight $a_e > 0$. The simplest case consists in constant weights

$$a_e = a \quad \text{for all } e \in E.$$

Edge-reinforced random walk

Edge-reinforced random walk is a stochastic process $(X_t)_{t \in \mathbb{N}_0}$ on G defined as follows:

- The process starts in a fixed vertex $0 \in V$: $X_0 = 0$
- At every time t it jumps to a nearest neigbor i of the current position X_t with probability proportional to the weight of the edge between X_t and i.
- Each time an edge is traversed, its weight is increased by one.

Edge-reinforced random walk - formal definition

Let $w_t(e)$ denote the weight of edge e at time t. We define $(X_t)_{t\in\mathbb{N}_0}$ and $(w_t(e))_{e\in E, t\in\mathbb{N}_0}$ simultaneously as follows:

- Initial weights: $w_0(e) = a_e$ for all $e \in E$
- Starting point: $X_0 = 0$
- Linear reinforcement:

$$w_t(e) = a_e + \sum_{s=0}^{t-1} 1_{\{X_s, X_{s+1}\}=e}, \quad t \in \mathbb{N}, e \in E.$$

- Probability of jump:

$$P(X_{t+1} = i | (X_s)_{0\le s\le t}) = \frac{w_t(\{X_t, i\})}{\sum_{e\in E: X_t\in e} w_t(e)} 1_{\{X_t, i\}\in E},$$

$t \in \mathbb{N}$, $i \in V$.

Linear reinforcement

The probability to jump to a neighboring point is proportional to the edge weight.

The reinforcement is linear in the number of edge crossings:

$$w_t(e) = a_e + k_t(e),$$

where

- $w_t(e) =$ weight of edge e at time t,
- $a_e =$ initial weight,
- $k_t(e) =$ number of traversals of edge e up to time t.

Motivation

- Edge-reinforced random walk was introduced by Persi Diaconis in 1986. He came up with the model when he was walking randomly through the streets of Paris and traversing the same streets over and over again.
- Othmer and Stevens used edge-reinforced random walk as a simple model for the motion of myxobacteria. These bacteria produce a slime and prefer to move on their slime trail.

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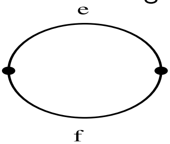
The Polya urn

Consider edge-reinforced random walk on the following graph:



The process of the edge weights $(w_t(e), w_t(f))_{t \in \mathbb{N}_0}$ behaves as follows:

- $w_0(e) = a$, $w_0(f) = b$
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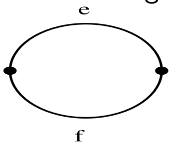

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- Consider an urn with a red and b blue balls.
- We draw a ball and return it to the urn with an additional ball of the same color.

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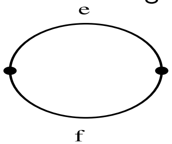

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- $w_0(e) = a$, $w_0(f) = b$
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This is a Polya urn process:

- Consider an urn with a red and b blue balls.
- We draw a ball and return it to the urn with an additional ball of the same color.
- $\left\{ \begin{matrix} w_t(e) \\ w_t(f) \end{matrix} \right\}$ corresponds to the number of $\left\{ \begin{matrix} \text{red} \\ \text{blue} \end{matrix} \right\}$ balls in the urn after t drawings.

An urn with polynomial reinforcement

- Consider an urn with a red and b blue balls.
- Let $\left\{ \begin{array}{c} k_t(e) \\ k_t(f) \end{array} \right\}$ denote the number of $\left\{ \begin{array}{c} \text{red} \\ \text{blue} \end{array} \right\}$ balls drawn from the urn up to time t. Set $\left\{ \begin{array}{c} w_t(e) = (a + k_t(e))^\alpha \\ w_t(f) = (b + k_t(f))^\alpha \end{array} \right\}$, where $\alpha > 0$ is fixed.
- The probability to draw a red ball at time t is given by

$$\frac{w_t(e)}{w_t(e) + w_t(f)}.$$

The urn with polynomial reinforcement

The probability to draw $k+1$ red balls at the beginning equals

$$\frac{a^\alpha}{a^\alpha + b^\alpha} \cdot \frac{(a+1)^\alpha}{(a+1)^\alpha + b^\alpha} \cdot \frac{(a+2)^\alpha}{(a+2)^\alpha + b^\alpha} \cdots \frac{(a+k)^\alpha}{(a+k)^\alpha + b^\alpha}.$$

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The probability to draw only red balls is given by

$$P(\text{only red}) = \prod_{i=0}^{\infty} \frac{(a+i)^\alpha}{(a+i)^\alpha + b^\alpha} = \prod_{i=0}^{\infty} \left(1 - \frac{b^\alpha}{(a+i)^\alpha + b^\alpha}\right).$$

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Hence $P(\text{only red}) > 0$ if and only if

$$\sum_{i=0}^{\infty} \frac{b^\alpha}{(a+i)^\alpha + b^\alpha} < \infty \quad \Longleftrightarrow \quad \sum_{i=1}^{\infty} \frac{1}{i^\alpha} < \infty \quad \Longleftrightarrow \quad \alpha > 1.$$

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In this sense, $\alpha = 1$ which corresponds to linear reinforcement is the critical case.

Random walk with superlinear edge-reinforcement

Random walk with superlinear edge-reinforcement is a stochastic process $(X_t)_{t\in\mathbb{N}_0}$ on a graph G defined as follows:

- Initial weights: a_e, $e \in E$
- Starting point: $X_0 = 0$
- $k_t(e)$ = number of traversals of edge e up to time t
- Superlinear reinforcement:

$$w_t(e) = (a_e + k_t(e))^\alpha, \quad t \in \mathbb{N}, e \in E$$

  for some $\alpha > 1$.
- Probability of jump:

$$P(X_{t+1} = i | (X_s)_{0\le s\le t}) = \frac{w_t(\{X_t, i\})}{\sum_{e\in E: X_t\in e} w_t(e)} 1_{\{X_t,i\}\in E},$$

  $t \in \mathbb{N}$, $i \in V$.

Random walk with superlinear edge-reinforcement

Theorem (Limic-Tarrès 2006, Cotar-Thacker 2016)

On any graph of bounded degree, random walk with superlinear edge-reinforcement gets stuck on one edge almost surely.
I.e. eventually, the random walk jumps back and forth on the same edge.

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On any graph of bounded degree, random walk with superlinear edge-reinforcement gets stuck on one edge almost surely. I.e. eventually, the random walk jumps back and forth on the same edge.

In particular, in the urn with superlinear reinforcement ($\alpha > 1$) we will eventually draw balls from the same color.

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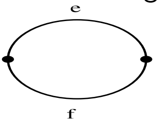
Exchangeability

Consider edge-reinforced random walk on the following graph:



with $w_0(e) = a$, $w_0(f) = b$.

Each time an edge is picked, its weight is increased by 1.

Let $Y_t \in \{e, f\}$ be the edge chosen by the random walk at time t.

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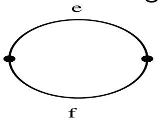

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Each time an edge is picked, its weight is increased by 1.

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Lemma

The sequence $(Y_t)_{t \in \mathbb{N}_0}$ is exchangeable: For all $n \in \mathbb{N}$ and any permutation π on $\{0, 1, \ldots, n\}$,

$$(Y_t)_{0 \le t \le n} \text{ and } (Y_{\pi(t)})_{0 \le t \le n} \text{ are equal in distribution.}$$

Moral: It does not matter in which order the edges are traversed, only the number of traversals is important.

Exchangeability - a proof

Let $n \in \mathbb{N}$, $y_t \in \{e, f\}$, $0 \leq t \leq n-1$,

$$k := |\{t \in \{0, \dots, n-1\} : y_t = e\}| = \text{ number of traversals of } e,$$
$$n - k = |\{t \in \{0, \dots, n-1\} : y_t = f\}| = \text{ number of traversals of } f.$$

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Then, the probability that the random walk chooses the edges y_t is given by

$$P(Y_t = y_t \, \forall 0 \leq t \leq n-1) = \frac{\prod_{t=0}^{k-1}(a+t) \prod_{t=0}^{n-k-1}(b+t)}{\prod_{t=0}^{n-1}(a+b+t)}.$$

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This probability depends only on the number of traversals of the edges, but not on the order of the y_t.

Asymptotic behavior

Lemma

Let

$$\alpha_n(e) := \frac{k_n(e)}{n}$$

be the proportion of crossings of edge e up to time n.

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be the proportion of crossings of edge e up to time n.
As $n \to \infty$ it converges almost surely to a random limit with a Beta(a, b)-distribution.

The Beta(a, b)-distribution has the density

$$\varphi_{a,b}(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}, \quad x \in (0,1).$$

For $a = b = 1$ this is the uniform distribution.

Asymptotic behavior - a rough idea of the argument

Using exchangeability, we have for $k \in \{0, \dots, n\}$

$$P\left(\alpha_n(e) = \frac{k}{n}\right) = \binom{n}{k} \frac{\prod_{t=0}^{k-1}(a+t) \prod_{t=0}^{n-k-1}(b+t)}{\prod_{t=0}^{n-1}(a+b+t)}.$$

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In the special case $a = b = 1$ this simplifies to

$$P\left(\alpha_n(e) = \frac{k}{n}\right) = \binom{n}{k} \frac{k!(n-k)!}{(n+1)!} = \frac{1}{n+1}.$$

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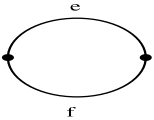
This can be used to prove weak convergence to a uniform distribution. For the almost sure convergence, one can use a martingale argument.

De Finetti's theorem: a mixture of i.i.d. processes



Theorem

The sequence of chosen edges is a mixture of i.i.d. sequences where the probability x to choose edge e is distributed according to a Beta(a, b)-distribution.

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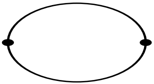

Theorem

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More formally: Let Q_x denote the law of an i.i.d. sequence where $\left\{ \begin{array}{c} e \\ f \end{array} \right\}$ is chosen with probability $\left\{ \begin{array}{c} x \\ 1-x \end{array} \right\}$. Then, one has for any event A

$$P((Y_t)_{t\in\mathbb{N}_0} \in A) = \int_0^1 Q_x((Y_t)_{t\in\mathbb{N}_0} \in A)\, \varphi_{a,b}(x)\, dx.$$

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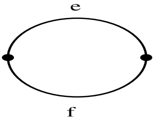

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This follows from de Finitti's theorem. It is not hard to check it directly.

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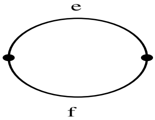

In particular, the probability to traverse edge e precisely k times up to time n is given by

$$P(k_n(e) = k) = \int_0^1 Q_x(k_n(e) = k)\, \varphi_{a,b}(x)\, dx$$
$$= \binom{n}{k} \int_0^1 x^k (1-x)^{n-k}\, \varphi_{a,b}(x)\, dx$$

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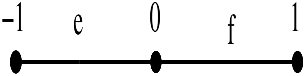
Three points in a line

Consider linearly edge-reinforced random walk on the following graph with $w_0(e) = a$, $w_0(f) = b$:



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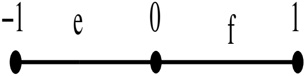

- When the random walk jumps from 0 to 1, it needs to return to 0 in the next step.
- When it returned to 0, the weight of f increased by 2.

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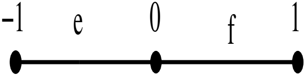

- When the random walk jumps from 0 to 1, it needs to return to 0 in the next step.
- When it returned to 0, the weight of f increased by 2.

Hence, the decision where to jump from 0 can be modelled by the following variant of a Polya urn:

- Consider an urn with a red and b blue balls.
- We draw a ball and return it to the urn with two additional balls of the same color.

The Polya urn where we add two balls

Let Polya(a, b, ℓ) denote the Polya urn process with

- initially a red and b blue balls,
- where in each step we return the ball together with ℓ balls of the same color.

The Polya urn where we add two balls

Let $\text{Polya}(a, b, \ell)$ denote the Polya urn process with

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$\text{Polya}(a, b, 2)$ and $\text{Polya}\left(\frac{a}{2}, \frac{b}{2}, 1\right)$ have the same distribution.

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Reason: The finite dimensional distributions agree, e.g.

$$P_{a,b,2}(Y_0 = e, Y_1 = e) = \frac{a}{a+b} \cdot \frac{a+2}{a+b+2} = \frac{\frac{a}{2}}{\frac{a+b}{2}} \cdot \frac{\frac{a}{2}+1}{\frac{a+b}{2}+1}$$

$$= P_{\frac{a}{2},\frac{b}{2},1}(Y_0 = e, Y_1 = e)$$

The Polya urn

More generally, for any $\ell > 0$,

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Hence, when we consider Polya$(a, b, 1)$, then $\left\{ \begin{array}{c} \text{small} \\ \text{large} \end{array} \right\}$ initial weights a, b correspond to $\left\{ \begin{array}{c} \text{strong} \\ \text{weak} \end{array} \right\}$ reinforcement.

Edge-reinforced random walk on $\mathbb{Z}$

Consider edge-reinforced random walk on $\mathbb{Z}$ starting at 0 with constant initial weights $a_e = a$ for all edges e.

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Assume the random walker is at $i \in \mathbb{Z}$ and it jumps from i to $i+1$. If it comes back to i at some later time, it comes back from the right and the weight of the edge $\{i, i+1\}$ has increased by 2.

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Decisions whether to go left or right are independent for different vertices.

Thus, we can put independent Polya urns at the vertices:

$$\begin{aligned}
\text{Polya}(a, a+1, 2) &\stackrel{d}{=} \text{Polya}\left(\tfrac{a}{2}, \tfrac{a+1}{2}, 1\right) && \text{at } i \leq -1, \\
\text{Polya}(a, a, 2) &\stackrel{d}{=} \text{Polya}\left(\tfrac{a}{2}, \tfrac{a}{2}, 1\right) && \text{at } i = 0, \\
\text{Polya}(a+1, a, 2) &\stackrel{d}{=} \text{Polya}\left(\tfrac{a+1}{2}, \tfrac{a}{2}, 1\right) && \text{at } i \geq 1,
\end{aligned}$$

In order to decide whether the random walk jumps left or right we draw a ball from the Polya urn.

Edge-reinforced random walk on $\mathbb{Z}$

Using that the Polya urn is a mixture of i.i.d. sequences, we conclude:

Lemma

Edge-reinforced random walk on $\mathbb{Z}$ has the same distribution as a random walk in a random environment where the environment is given by independent Beta-distributed jump probabilities.

Edge-reinforced random walk on $\mathbb{Z}$

More formally: For $p = (p_i)_{i \in \mathbb{Z}}$ with $p_i \in (0,1)$, let $Q_{0,p}$ denote the distribution of the Markovian random walk on $\mathbb{Z}$ starting at 0 with transition probabilities given by

$$
\begin{aligned}
Q_{0,p}(X_{t+1} = i+1 | X_t = i) =& p_i, \\
Q_{0,p}(X_{t+1} = i-1 | X_t = i) =& 1 - p_i,
\end{aligned}
$$

$i \in \mathbb{Z}$, $t \in \mathbb{N}_0$.

Edge-reinforced random walk on $\mathbb{Z}$

More formally: For $p = (p_i)_{i\in\mathbb{Z}}$ with $p_i \in (0,1)$, let $Q_{0,p}$ denote the distribution of the Markovian random walk on $\mathbb{Z}$ starting at 0 with transition probabilities given by

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$i \in \mathbb{Z}$, $t \in \mathbb{N}_0$. Let

$$\mu_{0,a} = \bigotimes_{i\in-\mathbb{N}} \mathrm{Beta}\left(\frac{a}{2}, \frac{a+1}{2}\right) \otimes \mathrm{Beta}\left(\frac{a}{2}, \frac{a}{2}\right) \bigotimes_{i\in\mathbb{N}} \mathrm{Beta}\left(\frac{a+1}{2}, \frac{a}{2}\right).$$

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The law of edge-reinforced random walk on $\mathbb{Z}$ is given by

$$P^{\mathrm{errw}}_{0,a}((X_t)_{t\in\mathbb{N}_0} \in A) = \int_{(0,1)^{\mathbb{Z}}} Q_{0,p}((X_t)_{t\in\mathbb{N}_0} \in A)\, \mu_{0,a}(dp)$$

for any event A.

Edge-reinforced random walk on $\mathbb{Z}$

Theorem

For all constant initial weights, edge-reinforced random walk on $\mathbb{Z}$ is recurrent. Even more, it is a unique mixture of positive recurrent Markov chains.

Edge-reinforced random walk on a binary tree

A similar construction can be done for any tree.

Pemantle used this to prove a phase transition for the binary tree.

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Theorem (Pemantle 1988)

There exists $a_c > 0$ such that edge-reinforced random walk on the binary tree with constant initial weights a has the following properties:

- *For $0 < a < a_c$, edge-reinforced random walk is recurrent. Almost all its paths visit every vertex infinitely often. Even more, it is a mixture of positive recurrent Markov chains.*
- *For $a > a_c$, edge-reinforced random walk is transient. Almost all its paths visit every vertex at most finitely often.*

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Partial exchangeability

Lemma

Edge-reinforced random walk is partially exchangeable:
The probability to traverse a finite path depends only on the starting point and on the number of crossings of the undirected edges.

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The following theorem is due to Diaconis-Freedman 1980.

Theorem (De Finetti's theorem for Markov chains)

If a process is partially exchangeable and it comes back to its starting point with probability one, then it is a mixture of reversible Markov chains.

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Theorem (De Finetti's theorem for Markov chains)

If a process is partially exchangeable and it comes back to its starting point with probability one, then it is a mixture of reversible Markov chains.

Using a Borel-Cantelli argument, one can verify the recurrence assumption for edge-reinforced random walk on any finite graph.

Reversible Markov chains

A Markov chain $(X_t)_{t\in\mathbb{N}_0}$ on V is reversible if it fulfills the detailed balance condition: there exists a reversible measure π such that for all $i, j \in V$ one has

$$\pi(i)p(i,j) = \pi(j)p(j,i),$$

where $p(i,j)$ denote the transition probabilities.

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An irreducible Markov chain is reversible if and only if it is a random walk on an undirected weighted graph: Put weight

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Thus, to describe the mixing measure for edge-reinforced random walk on a finite graph, we can describe a measure on edge weights x_e, $e \in E$.

Edge-reinforced random walk as a mixture

For $x = (x_e)_{e \in E} \in (0, \infty)^E$, let $Q_{0,x}$ denote the distribution of the random walk on the graph G with weights x_e on the undirected edges $e \in E$ starting at 0. I.e.

$$Q_{0,x}(X_{t+1} = i | (X_s)_{0 \le s \le t}) = \frac{x_{\{X_t, i\}}}{\sum_{e \in E: X_t \in e} x_e} 1_{\{X_t, i\} \in E},$$

$t \in \mathbb{N}$, $i \in V$.

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$t \in \mathbb{N}$, $i \in V$.

Theorem

For edge-reinforced random walk on any finite graph with any initial weights $a = (a_e)_{e\in E}$, there exists a unique probability measure $\mu_{0,a}$ on the set $(0,\infty)^E$ of edge weights such that for all events A, one has

$$P_{0,a}^{\mathrm{errw}}(A) = \int_{(0,\infty)^E} Q_{0,x}(A)\, \mu_{0,a}(dx).$$

Description of the mixing measure

- Let $e_0 \in E$ be a reference edge with $0 \in E_0$.
- d_v = vertex degree of v
- $x_v = \sum_{e \in E: v \in e} x_e$
- $\mathcal{T}$ = set of spanning trees of G

Theorem (Magic formula)

The mixing measure $\mu_{0,a}$ for the edge-reinforced random walk on a finite graph with constant initial weights a and starting point 0 is given by

$$\mu_{0,a}(dx) = \frac{1}{z} \frac{\sqrt{x_0} \prod_{e \in E} x_e^a}{\prod_{v \in V} x_v^{(a d_v + 1)/2}} \sqrt{\sum_{T \in \mathcal{T}} \prod_{e \in T} x_e} \, \delta_1(dx_{e_0}) \prod_{e \in E \setminus \{e_0\}} \frac{dx_e}{x_e}$$

with a normalizing constant z and dx_e the Lebesgue measure on $(0, \infty)$.

The mixing measure

The mixing measure was described explicitly by

- [Coppersmith-Diaconis, 1986] (The first paper about reinforced random walks, unpublished.)
- [Keane-R., 2000] (The first paper of my Ph.D. thesis.)
- [Merkl-Öry-R., 2008]
- [Sabot-Tarrès-Zeng 2016]
- ...

It is called "Magic formula". The name is due to Janos Engländer.

Consequences of the mixure of Markov chains

- The dependence structure of the edge weights in the magic formula is not easy.
- It took almost 20 years before the magic formula was used to prove results about edge-reinforced random walks.

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- It took almost 20 years before the magic formula was used to prove results about edge-reinforced random walks.

Finally, it enabled proofs of many results, among others, recurrence and asymptotic properties of the process

- for $\mathbb{Z} \times G$ with a finite graph G and arbitrary constant initial weights [Merkl & R., 2005-2009],
- for a diluted version of $\mathbb{Z}^2$ with small initial weights [Merkl & R., 2009].

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Results for ladders

Consider edge-reinforced random walk on $\mathbb{Z} \times G$ with a finite graph G with constant initial weights.

Theorem (Merkl & R. 2008)

Edge-reinforced random walk on $\mathbb{Z} \times G$ is recurrent. Even more, it is a unique mixture of positive recurrent Markov chains.

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Consider edge-reinforced random walk on $\mathbb{Z} \times G$ with a finite graph G with constant initial weights.

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Edge-reinforced random walk on $\mathbb{Z} \times G$ is recurrent. Even more, it is a unique mixture of positive recurrent Markov chains.

- Let μ denote the mixing measure.
- For $i \in V$, let $x_i = \sum_{e \in E: i \in e} x_e$

Theorem (Merkl & R. 2008)

There exists a constant $c > 0$ such that for μ-almost all x one has

$$x_i \leq x_0 \exp(-c|i|)$$

for all but finitely many $i \in V$.

Results for ladders

Theorem (Merkl & R. 2008)

There exist constants $c_1, c_2, c_3 > 0$ such that the following hold for edge-reinforced random walk on $\mathbb{Z} \times G$ with constant initial weights.

For all $t \in \mathbb{N}_0$ and all $i \in V$, one has

$$P_{0,a}^{\mathrm{errw}}(X_t = i) \leq c_1 e^{-c_2|i|}.$$

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$$P_{0,a}^{\mathrm{errw}}\left(\max_{0 \leq s \leq t} |X_s| \leq c_3 \log t \text{ for all but finitely many } t\right) = 1$$

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$$P_{0,a}^{\mathrm{errw}}(\tau_i < \tau_0) \leq c_1 e^{-c_2|i|}$$

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Connection with the vertex-reinforced jump process

In 2011 Sabot and Tarrès found a connection between edge-reinforced random walk and the vertex-reinforced jump process which turned out to be very useful.

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In 2011 Sabot and Tarrès found a connection between edge-reinforced random walk and the vertex-reinforced jump process which turned out to be very useful.

- Consider a locally finite, undirected graph $G = (V, E)$ with edge weights $W_e > 0$, $e \in E$.
- The vertex-reinforced jump process $Y = (Y_t)_{t \geq 0}$ is a process in continuous time where given $(Y_s)_{s \leq t}$ the particle jumps from site i to a neighbor j with rate

  $$W_{ij} L_j(t),$$

  where

  $$L_j(t) = 1 + \int_0^t 1_{\{Y_s = j\}} \, ds$$

  is the local time at j with offset 1.

The vertex-reinforced jump process as a mixture

Theorem (Sabot-Tarrès 2011)

On any finite graph, the discrete-time process $\tilde{Y}$ *associated with the vertex-reinforced jump process is a mixture of reversible Markov chains.*

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On any finite graph, the discrete-time process $\tilde{Y}$ associated with the vertex-reinforced jump process is a mixture of reversible Markov chains.

There is a unique probability measure $\mathbb{P}_0^W$ on $(0,\infty)^E$, depending on the starting point 0 and the weights $W=(W_e)_{e\in E}$ of the vertex-reinforced jump process such that for any event $A\subseteq V^{\mathbb{N}_0}$, one has

$$P_{0,W}^{\mathrm{vrjp}}(\tilde{Y}\in A)=\int_{(0,\infty)^E} Q_{0,x}(A)\,\mathbb{P}_0^W(dx).$$

The mixing measure for the vertex-reinforced jump process

Theorem (Sabot-Tarrès 2011)

The mixing measure $\mathbb{P}_0^W$ *can be described by putting on the edge* $\{i, j\}$ *the weight*

$$W_{ij} e^{u_i + u_j}$$

with $(u_i)_{i \in V}$ *distributed according to (a marginal of) Zirnbauer's supersymmetric (susy) hyperbolic non-linear sigma model.*

The supersymmetric hyperbolic non-linear sigma model was introduced by Zirnbauer in 1991 in a completely different context.

The supersymmetric hyperbolic non-linear sigma model

- Zirnbauer writes that it may serve as a toy model for studying diffusion and localization in disordered one-electron systems.
- It is a statistical mechanics model with a Hamiltonian like in the Ising model except that the spin variables are much more complicated.
- It is tractable because of its (super-)symmetries.

A new representation of the mixing measure for edge-reinforced random walk

Theorem (Sabot-Tarrès 2011)

On any finite graph, the edge-reinforced random walk X is a mixture of the law of the discrete-time process $\tilde{Y}$ associated to the vertex-reinforced jump process if one takes W_e, $e \in E$, independent and Gamma(a_e)-distributed.

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Then, for any event $A \subseteq V^{\mathbb{N}_0}$, one has

$$P_{0,a}^{\mathrm{errw}}(X \in A) = \int_{(0,\infty)^E} P_{0,W}^{\mathrm{vrjp}}(\tilde{Y} \in A) \prod_{e \in E} \Gamma_{a_e}(dW_e)$$

$$= \int_{(0,\infty)^E} \int Q_{0,(W_{ij} e^{u_i + u_j})_{\{i,j\} \in E}}(A)\, \mu_0^{W,\mathrm{susy}}(du) \prod_{e \in E} \Gamma_{a_e}(dW_e),$$

where $\mu_0^{W,\mathrm{susy}}$ denotes the law of Zirnbauer's model.

Consequences for edge-reinforced random walk

This connection allowed to transfer results from the susy model to edge-reinforced random walk.

Consider edge-reinforced random walk on $\mathbb{Z}^d$ with constant initial weights. There is a phase transition between recurrence and transience.

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  recurrence for $d \geq 2$ for small initial weights
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[Angel-Crawford-Kozma 2012]
gave an alternative proof for the recurrence part without using the connection to the non-linear supersymmetric sigma model.

Recurrence of edge-reinforced random walk on $\mathbb{Z}^2$

Theorem (Sabot-Zeng 2015)

On $\mathbb{Z}^2$, edge-reinforced random walk is recurrent for all constant initial weights.

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The proof is not easy.

Key ingredients:

- a martingale
- an estimate from [Merkl & R., 2008]:

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Key ingredients:

- a martingale
- an estimate from [Merkl & R., 2008]:
  Let τ_i denote the first hitting time of i. Then, there exists $\alpha > 0$ such that for all $i \in \mathbb{Z}^2$

$$P_{0,a}^{\mathrm{errw}}(\tau_i < \tau_0) \leq \|i\|_\infty^{-\alpha}.$$

Estimate for the hitting probability

There exists $\alpha > 0$ such that for all $i \in \mathbb{Z}^2$

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Estimate for the hitting probability

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Let $B_n = [-n, n]^2 \cap \mathbb{Z}^2$.

The probability to hit the boundary of B_n before returning to the origin for the edge-reinforced random walk is given by

$$P_{0,a}^{\mathrm{errw}}(\tau_{\partial B_n} < \tau_0) \leq \sum_{i \in \partial B_n} P_{0,a}^{\mathrm{errw}}(\tau_i < \tau_0) \leq cn \cdot n^{-\alpha}$$

with a constant c.

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For recurrence one needs

$$\lim_{n\to\infty} P_{0,a}^{\text{errw}}(\tau_{\partial B_n} < \tau_0) = 0.$$

This is garanteed only for $\alpha > 1$, which is not known.
However, the argument of Sabot and Tarrès worked with $\alpha > 0$.
They needed decay of the weights to get a contradiction.

Method of proof

It is crucial that we have a mixture of reversible Markov chains.

Consider the Markovian random walk with law $Q_{0,x}$.
A reversible measure is given by

$$\pi_i = \sum_{e \in E : i \in e} x_e.$$

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If we can show that the edge weights are summable

$$\sum_{e \in E} x_e < \infty \Rightarrow \sum_{i \in V} \pi_i < \infty$$

the random walk is positive recurrent.

Decay of the weights gives also bounds on the escape probability of the random walk.

Method of proof

Hard part of the proof: Bound the edge weights.

- for ladders: transfer operator
- symmetry for finite pieces with periodic boundary conditions
- Best method nowadays: use the supersymmetric sigma model.

Thank you for your attention!