Hydrodynamic limit in the Hyperbolic Space-Time Scale

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- Corresponding to different paramenters there are different *partial* equilibriums:
  - *mechanical equilibrium*: constant pressure or tension profiles,
  - *thermal equilibrium*: constant temperature profiles.
- These partial equilibriums may be reached at different time scales: *typically* mechanical equilibrium is reached faster than thermal equilibrium.

Mechanical and Thermal equilibrium

- **Mechanical Equilibrium** is reached in **hyperbolic** time scales (same rescaling of space and time), and is driven by Euler system of equations (for a compressible gas). It involves the ballistic evolution of the long waves (mechanical modes).

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- When thermal conductivity is finite, **Thermal Equilibrium** is reached later, in the **diffusive** time scales ($\text{time}^2 = \text{space}$), and temperature (or thermal energy) profiles evolve following *heat equation*.
- If thermal conductivity is infinite, **Thermal Equilibrium** is reached in a **super-diffusive** time scales ($\text{time}^\alpha = \text{space}, \alpha < 2$), and typically temperature (or thermal energy) profiles evolve following a *fractional heat equation*.

Boundary Conditions

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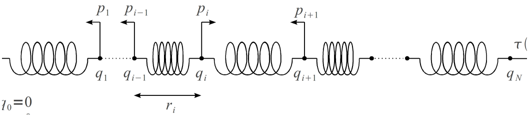
Most of non-equilibrium situation are obtained by

- changing boundary conditions in time
- applying boundary conditions corresponding to different equilibrium states, obtaining dynamics that have *non-equilibrium stationary states* (NESS).

Chain of oscillators



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\dot{r}_x(t) &= p_x(t) - p_{x-1}(t), && x = 1, \dots, N \\
\dot{p}_x(t) &= V'(r_{x+1}(t)) - V'(r_x(t)) && x = 1, \dots, N-1 \\
\dot{p}_N(t) &= \tau(t/N) - V'(r_N(t)) \\
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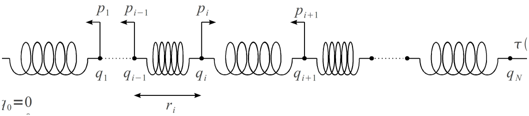

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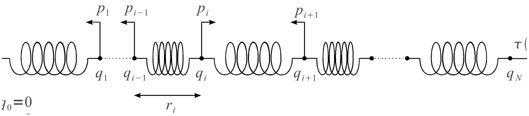

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We are interested in the *macroscopic* evolution of $(r_x(t), p_x(t), \mathcal{E}_x(t))$.

Gibbs measures and Thermodynamic Entropy

For $\tau(t) = \tau$ constant in time, a class of stationary measures is given by the Gibbs measures at temperature β^{-1}, tension τ

$$d\mu_{\beta,\tau,p} = \prod_{x=1}^{N} e^{-\beta(\mathcal{E}_x - \tau r_x) - \mathcal{G}(\beta,\tau)} dp_x dr_x$$

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Thermodynamic entropy is

$$S(u,r) = \inf_{\tau,\beta} \{-\beta\tau r + \beta u - \mathcal{G}(\beta,\tau)\}$$

$$\beta(u,r) = \partial_u S(u,r), \qquad \tau(u,r) = -\beta^{-1}\partial_r S(u,r).$$

Ergodicity (of the infinite system)

Consider the corresponding infinite dynamics:

$$\begin{aligned} \dot{r}_x(t) &= p_x(t) - p_{x-1}(t), \\ \dot{p}_x(t) &= V'(r_{x+1}(t)) - V'(r_x(t)) \qquad\qquad x \in \mathbb{Z} \end{aligned}$$

Theorem

(Fritz, Funaki, Lebowitz, PTRF 1994) Assume that a probability ν is translation invariant, stationary, finite entropy density, and the conditional measure $\nu(dp|r)$ is exchangeable.
Then ν is a convex combination of Gibbs measures $d\mu_{\beta,\tau,\mathbf{p}}$.

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- Chaoticity of the dynamics, due to the non-linearity of V, should give such ergodic property
- Adding conservative noise (stochastic collisions) to the dynamics one obtain ergodicity.

Hyperbolic Scaling, Euler equations

3 conserved quantities: we expect the weak convergence to the hyperbolic system of PDE

$$\frac{1}{N}\sum_x G(x/N)\begin{pmatrix} r_x(Nt) \\ p_x(Nt) \\ \mathcal{E}_x(Nt)\end{pmatrix} \underset{N\to\infty}{\longrightarrow} \int_0^1 G(y)\begin{pmatrix} r(y,t) \\ p(y,t) \\ \mathfrak{e}(y,t)\end{pmatrix} dy$$

$$\begin{aligned}
\partial_t r(t,y) &= \partial_y p(t,y) \\
\partial_t p(t,y) &= \partial_y \tau[u(t,y), r(t,y)] \\
\partial_t \mathfrak{e}(t,y) &= \partial_y(\tau[u(t,y), r(t,y)]p(t,y))
\end{aligned}$$

where $u = \mathfrak{e} - p^2/2$: internal energy.
and, *for smooth solutions*, the boundary conditions:

$$p(t,0) = 0, \qquad \tau[u(t,1), r(t,1)] = \tau(t)$$

Euristics

take $G:[0,1]\to\mathbb{R}$ with compact support in $(0,1)$,

$$\frac{d}{dt}\frac{1}{N}\sum_x G(x/N)\begin{pmatrix} r_x(Nt)\\ p_x(Nt)\\ \mathcal{E}_x(Nt)\end{pmatrix} = \sum_x G(x/N)\begin{pmatrix} \nabla p_{x-1}(Nt)\\ \nabla V'(r_x(Nt))\\ \nabla\left[p_x(Nt)V'(r_x(Nt)\right]\end{pmatrix}$$

$$\sim -\frac{1}{N}\sum_x G'(x/N)\begin{pmatrix} p_x(Nt)\\ V'(r_x(Nt))\\ p_x(Nt)V'(r_x(Nt)\end{pmatrix}$$

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assuming local equilibrium, we have

$$\sim -\int_0^1 G'(y)\begin{pmatrix} p(t,y)\\ \tau(u(t,y),r(t,y))\\ p(t,y)\tau(u(t,y),r(t,y))\end{pmatrix}dy$$

Note that $y\in[0,1]$ is the *material (Lagrangian)* coordinate.

Results with conservative stochastic dynamics

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- To prove some form of *local equilibrium* we need to add stochastic terms to the dynamics (the deterministic non-linear case is too difficult).
- Random exchanges of velocities between nearest neighbor particles, conserve energy, momentum and volume, destroying all other (possible) conservation laws. It provides the *right ergodicity* property.
- With such noise in the dynamics, for smooth solutions the HL is proven in:
  - N. Even, S.O., ARMA (2014) (with boundary conditions),
  - S.O., SRS Varadhan, HT Yau, CMP (1993) (periodic bc).

Harmonic Oscillators Chain

This is an example of a non-ergodic dynamics:

$$V(r) = r^2/2$$

in fact it is a *completely integrable dynamics*:

$$\dot{q}_x = p_x, \qquad \dot{p}_x = \Delta q_x = q_{x+1} + q_{x-1} - q_x,$$

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Take here $x = 1, \dots, N$,

$$\hat{f}(k) = \sum_x f_x e^{i2\pi kx} \qquad k \in \{0, 1/N, \dots, (N-1)/N\}$$

$\omega(k) = 2|\sin(\pi k)|$ dispersion relation:

$$\mathcal{H} = \sum_x \mathcal{E}_x = \frac{1}{2N} \sum_k \left[\omega(k)^2 |\hat{q}(k)|^2 + |\hat{p}(k)|^2\right]$$

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$$\frac{d}{dt}\hat{\psi}(t,k) = -i\omega(k)\hat{\psi}(t,k) \qquad \hat{\psi}(t,k) = e^{-i\omega(k)t}\hat{\psi}(0,k)$$

Harmonic Oscillators Chain: Quantum Dynamics

$$p_x = -i\partial_{q_x} = -i\left(\partial_{r_{x+1}} - \partial_{r_x}\right)$$

$$\mathcal{E}_x = \frac{1}{2}\left(p_x^2 + r_x^2\right)$$

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Heisenber evolution $\frac{d}{dt}A(t) = i[\mathcal{H}, A(t)]$

$$a_k(t) = e^{-i\omega(k)t}a_k, \qquad a_k^*(t) = e^{-i\omega(k)t}a_k^*.$$

Harmonic Chain: Thermal Equilibrium (Classic case)

Consider the chain in *thermal* equilibrium: initial distribution with covariances

$$\langle\, r_x(0); r_{x'}(0) \,\rangle = \langle\, p_x(0); p_{x'}(0) \,\rangle = \beta^{-1}\delta_{x,x'}, \qquad \langle\, q_x; p_{x'} \,\rangle = 0,$$

for some inverse temperature β, while in *mechanical local equilibrium*:

$$\langle\, r_{[Ny]}(0) \,\rangle \longrightarrow r(0,y), \qquad \langle\, p_{[Ny]}(0) \,\rangle \longrightarrow p(0,y).$$

Harmonic Chain: Thermal Equilibrium (classic case)

thermal equilibrium is conserved by the dynamics: for any $t \geq 0$

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Proof.

Thermal equilibrium is Fourier space is:

$$\langle\, \hat{\psi}(k,0)^*; \hat{\psi}(k',0) \,\rangle = 2\beta^{-1}\delta(k-k'), \qquad \langle\, \hat{\psi}(k,0); \hat{\psi}(k',0) \,\rangle = 0.$$

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Consequently

$$\langle\, \hat\psi(k,t)^*; \hat\psi(k',t) \,\rangle = e^{i(\omega(k)-\omega(k'))t} \langle\, \hat\psi(k,0)^*; \hat\psi(k',0) \,\rangle = 2\beta^{-1}\delta(k-k'$$
$$\langle\, \hat\psi(k,t); \hat\psi(k',t) \,\rangle = e^{-i(\omega(k)+\omega(k'))t} \langle\, \hat\psi(k,0); \hat\psi(k',0) \,\rangle = 0.$$

$\square$

Harmonic Chain: Thermal Equilibrium implies Euler Equation limit

$r_{[Ny]}(Nt)$ and $p_{[Ny]}(Nt)$ converge weakly to the solution of the linear wave equation

$$\partial_t \mathbf{r}(y,t) = \partial_y \mathbf{p}(y,t), \qquad \partial_t \mathbf{p}(y,t) = \partial_y \mathbf{r}(y,t).$$

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For the energy, because of the thermal equilibrium, for any $t \geq 0$:

$$\langle \mathcal{E}_x(t) \rangle = \beta^{-1} + \frac{1}{2}\left(\langle p_x(t) \rangle^2 + \langle r_x(t) \rangle^2\right)$$

$$\langle \mathcal{E}_{[Ny]}(Nt) \rangle \longrightarrow \mathbf{e}(y,t) = \beta^{-1} + \frac{1}{2}\left(\mathbf{p}^2(y,t) + \mathbf{r}^2(y,t)\right),$$

$$\partial_t \mathbf{e}(y,t) = \partial_y \left(\mathbf{p}(y,t)\,\mathbf{r}(y,t)\right).$$

Initial density matrix ρ_β, define

$$\langle\, A \,\rangle = \mathrm{tr}\left(A\rho_\beta\right)), \qquad \langle\, A; B \,\rangle = \langle\, AB \,\rangle - \langle\, A \,\rangle\langle\, B \,\rangle$$

such that

$$\langle\, r_x(0); r_{x'}(0) \,\rangle = \langle\, p_x(0); p_{x'}(0) \,\rangle = C_\beta(x-x'), \qquad \langle\, q_x; p_{x'} \,\rangle = \frac{i}{2}\delta(x-x')$$

$$C_\beta(x) = \frac{1}{N}\left[\beta^{-1} + \sum_{k\neq 0} e^{2\pi ikx}\left(\frac{\omega_k}{e^{\beta\omega_k}-1} + \frac{\omega_k}{2}\right)\right] \tag{1}$$

Quantum Harmonic Chain: Thermal Equilibrium

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$$\langle\, r_{[Ny]}(0) \,\rangle \longrightarrow r(0,y), \qquad \langle\, p_{[Ny]}(0) \,\rangle \longrightarrow p(0,y).$$

$$\langle\, \mathcal{E}_{[Ny]} \,\rangle \longrightarrow \mathbf{e}\,(y) = \bar{C}(\beta) + \frac{1}{2}\left(\mathbf{p}^2(y) + \mathbf{r}^2(y)\right),$$

$$\bar{C}(\beta) = \int_0^1 \omega(k)\left(\frac{1}{e^{\beta\omega(k)}-1} + \frac{1}{2}\right) dk \underset{\beta\to 0}{\sim} \beta^{-1}$$

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$$\partial_t \mathbf{e}(y,t) = \partial_y\left(\mathbf{p}(y,t)\,\mathbf{r}(y,t)\right).$$

Harmonic Chain: Local Thermal Equilibrium is not conserved

The argument fails dramatically if the system is not in thermal equilibrium, even local thermal Gibbs

$$\langle\, r_x(0); r_{x'}(0)\, \rangle = \langle\, p_x(0); p_{x'}(0)\, \rangle = \beta^{-1}\left(\frac{x}{N}\right)\delta_{x,x'}, \quad \langle\, q_x(0); p_{x'}(0)\, \rangle = 0 \tag{2}$$

is not conserved, and correlations between $p_x(t)$ and $r_x(t)$ build up in time.

No autonomous macroscopic equation for the energy!

There are infinite many conservation laws.

Wigner distribution

$\xi \in \mathbb{R},\ k \in [0,1]$,

$$\widehat{W}_N(\xi,k,t) \ := \ \frac{2}{N}\left\langle\!\!\left\langle \hat{\psi}^*\left(Nt, k - \frac{\xi}{2N}\right)\hat{\psi}\left(Nt, k + \frac{\xi}{2N}\right)\right\rangle\!\!\right\rangle$$

$$W_N(y,k,t) = \int \widehat{W}_N(t,\eta,k) e^{-i2\pi\xi y}\, d\eta, \qquad y \in \mathbb{R},$$

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$$W_N(y, k, t) = \int \widehat{W}_N(t, \eta, k) e^{-i2\pi\xi y} \, d\eta, \qquad y \in \mathbb{R},$$

In the limit it decompose in a thermal and a mechanical part:

$$\lim_{N \to \infty} \widehat{W}_N(\xi, k, t) = \widehat{W}_{th}(\xi, k, t) + \widehat{W}_m(\xi, t) \, \delta_0(dk) \tag{3}$$

The mechanical part $\widehat{W}_m(\xi, t)$ is the Fourier transform of the mechanical energy

$$\widehat{W}_m(\xi, t) = \int \frac{1}{2} \left(\mathbf{p}^2(y, t) + \mathbf{r}^2(y, t) \right) e^{i2\pi\xi y} \, dy,$$

Wigner distribution

For the thermal Wigner distribution it holds the transport equation:

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in fact for $k \neq 0$

$$\begin{aligned} \widehat{W}_N(\xi,k,t) \; &:= \; e^{i\left[\omega\left(k-\frac{\xi}{2N}\right)-\omega\left(k+\frac{\xi}{2N}\right)\right]Nt}\,\widehat{W}_N(\xi,k,0) \\ &\underset{N\to\infty}{\sim} \; e^{-i\omega'(k)\xi t}\,\widehat{W}_{th}(\xi,k,0) \end{aligned}$$

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$$W(t,y,k) = W(0, y - \frac{\omega'(k)}{2\pi}t, k)$$

Phonon of wave number k moves freely with velocity $\frac{\omega'(k)}{2\pi}$.

Wigner distribution

Consequently the thermal energy $\tilde{\mathbf{e}}(y,t)$ (i.e. temperature) evolves non autonomously following the equation

$$\partial_t \tilde{\mathbf{e}}(y,t) + \partial_y J(y,t) = 0, \qquad J(y,t) = \int \omega'(k) W_{th}(y,k,t)\, dk.$$

We say that the system is in *local equilibrium* if $W_{th}(y,k) = \beta^{-1}(y)$ constant in k.
Starting in thermal equilibrium means $W_{th}(y,k,0) = \beta^{-1}$ and trivially $W_{th}(y,k,t) = \beta^{-1}$ for any $t > 0$.
But starting with local equilibrium, i.e. $W(y,k,0) = \beta^{-1}(y)$ constant in k, we have a non autonomous evolution of $\tilde{e}(y,t)$.

Harmonic Chain with Random Masses

The problem with the harmonic chain is that thermal waves of wavenumber k *move* with speed $\omega'(k)$, if they are not uniformed distributed (i.e. the system is not in thermal equilibrium), the temperature profile will not remain constant, as it should be following the Euler equations.

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If the masses are random, the thermal modes remains localized (frozen), by Anderson localization. This allows to close the energy equation, **without local equilibrium**.

Harmonic Chain with Random Masses

(F. Huveneers, C. Bernardin, S.Olla, 2017)

$\{m_x\}$ i.i.d. with absolutely continuous distribution,
$0 < m_- \leq m_x \leq m_+$,
$\overline{m} = \mathbb{E}(m_x)$.

$$m_x \dot{q}_x(t) = p_x(t), \qquad \dot{p}_x(t) = \Delta q_x(t), \qquad x = 1, \dots, N$$

with $q_0 = q_1$ and $q_{N+1} = q_N$ as boundary conditions.

Gibbs States, Local Gibbs States

The Gibbs states are characterized by three parameters: $\beta > 0$ and $p, r \in \mathbb{R}$. Its probability density writes

$$\prod_{x=1}^{N} \frac{e^{-\frac{\beta m_x}{2}\left(\frac{p_x}{m_x} - \frac{p}{\bar{m}}\right)^2 - \frac{\beta}{2}(r_x - r)^2}}{Z(\beta, p, r, m_x)}.$$

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We start with a local Gibbs state:

$$\prod_{x=1}^{N} \frac{e^{-\frac{\beta(x/N)m_x}{2}\left(\frac{p_x}{m_x}-\frac{p(x/N)}{\overline{m}}\right)^2-\frac{\beta(x/N)}{2}(r_x-r(x/N))^2}}{Z(\beta(x/N), p(x/N), r(x/N), m_x)}.$$

Harmonic Chain with Random Masses: hydrodynamic limit

Almost surely with respect to $\{m_x\}$:

$$< r_{[Ny]}(Nt) >, < p_{[Ny]}(Nt) >, < \mathcal{E}_{[Ny]}(Nt) > \rightharpoonup (\mathbf{r}(y,t), \mathbf{p}(y,t), \mathfrak{e}(y,t))$$

$$\begin{aligned}
\partial_t \mathbf{r}(t,y) &= \frac{1}{\overline{m}} \partial_y \mathbf{p}(t,y) \\
\partial_t \mathbf{p}(t,y) &= \partial_y \mathbf{r}(t,y) \\
\partial_t \mathfrak{e}(t,y) &= \frac{1}{\overline{m}} \partial_y \left(\mathbf{r}(t,y)\mathbf{p}(t,y)\right)
\end{aligned}$$

with initial conditions:

$$\mathbf{r}(y,0) = r(y), \qquad \mathbf{p}(y,0) = p(y), \qquad \mathfrak{e}(y,0) = \frac{1}{\beta(y)} + \frac{p^2(y)}{2\overline{m}} + \frac{r^2(y)}{2}.$$

Random Masses: Localization of Thermal Modes

Equation of motion can be written as

$$\ddot{r}_x = -(\nabla^* M^{-1} \nabla r)_x \quad (1 \le x \le N-1), \qquad \ddot{p}_x = \left(\Delta M^{-1} p\right)_x \quad (1 \le x \le N),$$

where $M_{x,x'} = \delta_{x,x'} m_x$.

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$$M^{-1/2}(-\Delta) M^{1/2} \varphi^k = \omega_k^2 \, \varphi^k, \qquad k = 0, \dots, N-1.$$

$$\psi^k = M^{-1/2} \varphi^k, \qquad M^{-1} \Delta \psi^k = \omega_k^2 \psi_k$$

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$$r(t) = \sum_{k=1}^{N-1} \left(\frac{\langle \nabla \psi^k, r(0) \rangle}{\omega_k} \cos \omega_k t + \langle \psi^k, p(0) \rangle \sin \omega_k t \right) \frac{\nabla \psi^k}{\omega_k},$$

$$p(t) = \sum_{k=0}^{N-1} \left(\langle \psi^k, p(0) \rangle \cos \omega_k t - \frac{\langle \nabla \psi^k, r(0) \rangle}{\omega_k} \sin \omega_k t \right) M \psi^k.$$

Localization of Thermal Modes

Localization length ξ_k diverges with N:

$$\xi_k^{-1} \sim \omega_k^2 \sim \left(\frac{k}{N}\right)^2,$$

only the modes $k > \sqrt{N}$ are localized.

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only the modes $k > \sqrt{N}$ are localized.
More precisely: for $0 < \alpha < \frac{1}{2}$

$$\mathbb{E}\left(\sum_{k=N^{1-\alpha}}^{N-1} |\psi_x^k \psi_{x'}^k|\right) \leq Ce^{-cN^{-2\alpha}|x-x'|}.$$

This estimate is enough to prove that thermal modes remains localized and do not *move* macroscopically.

Random masses: Larger time scales

Assume for simplicity that we are in a *mechanical equilibrium*:

$$\langle\ r_x(0)\ \rangle = 0, \qquad \langle\ p_x(0)\ \rangle = 0,$$

(only thermal energy present)
but not in thermal equilibrium, then, for any $\alpha \geq 1$

$$< \mathcal{E}_{[Ny]}(N^\alpha t) > \underset{N\ to\infty}{\longrightarrow} \mathbf{e}(0, y) = \bar{C}(\beta(y))$$

NO evolution for the temperature profile at any scale!

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NO evolution for the temperature profile at any scale!
In particular, for $\alpha = 2$ (diffusive scaling), thermal diffusivity is null.

Open questions for the quantum case

- In order to deal with the anharmonic interaction, in the classical case, conservative noise is added to obtain ergodicity of the infinite dynamics (unique characterization of the translational invariant stationary states)
  (cf B. Nachtergaele, and H-T Yau, CMP 2003).
  How to add *conservative noise* in the quantum dynamics in order to obtain similar result?

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- In order to deal with the anharmonic interaction, in the classical case, conservative noise is added to obtain ergodicity of the infinite dynamics (unique characterization of the translational invariant stationary states)
(cf B. Nachtergaele, and H-T Yau, CMP 2003).
How to add *conservative noise* in the quantum dynamics in order to obtain similar result?
- Boundary tension? More generally boundary conditions, thermostat etc.

entropy evolution

$$\boxed{\begin{array}{ll} \partial_t r = \partial_x p \qquad \partial_t p = \partial_x \tau \qquad \partial_t \mathfrak{e} = \partial_x(\tau p) \\ p(t,0) = 0, \qquad \tau(r(1,t), u(1,t)) = \tau(t) \end{array}}$$

$U = \mathfrak{e} - p^2/2$, $\beta = \frac{\partial S}{\partial U}$, $\tau = -\frac{1}{\beta}\frac{\partial S}{\partial r}$

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$U = \mathfrak{e} - p^2/2$, $\beta = \frac{\partial S}{\partial U}$, $\tau = -\frac{1}{\beta}\frac{\partial S}{\partial r}$

For smooth solutions:

$$\begin{aligned} \frac{d}{dt} S(u(y,t), r(y,t)) &= \beta\left(\partial_t e - p\partial_t p\right) - \beta\tau\partial_t r \\ &= \beta\left(\partial_x(\tau p) - p\partial_x\tau - \tau\partial_x p\right) = 0 \end{aligned}$$

The evolution is *isoentropic* in the smooth regime.

Shocks, contact discontinuities, weak solutions, entropy solutions

Even starting with initial smooth profiles, hyperbolic non-linear systems develops discontinuities:

- shocks: discontinuities in the tension profile,

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- entropy solutions
- viscosity solutions

weak solutions

Consider a hyperbolic system of conservation laws

$$v_t + f(v)_x = 0,$$

a weak solution $v(t,y)$ on an open set $\Omega \subset \mathbb{R}^2$ satisfies, for any function $\phi(t,y) \in \mathcal{C}^1$ with compact support in Ω

$$\iint_\Omega [\phi_t v + \phi_y f(v)]\, dy\, dt = 0$$

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No continuity assumption is made on v.
In the Euler case, $v = (r, p, e)$, $u = e - p^2/2$ and

$$f(v) = -\begin{pmatrix} p \\ \tau(u, r) \\ p\tau(u, r) \end{pmatrix}$$

Strictly Hyperbolic System: the Jacobian matrix Df has real distinct eigenvalues.

weak solutions: Cauchy initial problem

A weak solution of

$$v_t + f(v)_x = 0, \qquad v(0,y) = v_0(y),$$

is a weak solution of the Cauchy initial data problem
if $t \in [0, T] \to v(t,\cdot)$ is continuous in L^1_{loc}.

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unfortunately it may not be unique!

Existence proved only for v_0 of bounded variation (Glimm,....).

Entropic weak solutions

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$$D\eta(v) \cdot Df(v) = Dq(v)$$

that implies for smooth solutions:

$$\eta(v)_t + q(v)_x = 0.$$

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- For the Euler System: the thermodynamic entropy $\eta(v) = S(e - p^2/2, r)$ is an *entropy function*.

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as distribution, for *any* entropy pair (η, q).
This implies that total entropy $\int \eta(v(t,y))dy$ increase in time (with no b.c. here).
Existence is proven only under *bounded variation* initial conditions.
The conjecture is that entropy-admissible solutions are *unique*.

Vanishing viscosity solutions

$$v_t^\varepsilon + f(v^\varepsilon)_x = \varepsilon v_{xx}^\varepsilon,$$

or more general

$$v_t^\varepsilon + f(v^\varepsilon)_x = \varepsilon \Lambda(v^\varepsilon),$$

where Λ is a second order differential operator (eventually non-linear).

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If v^ε converges in L^1_{loc} as $\varepsilon \to 0^+$, this is called a *vanishing viscosity solution*.

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Bianchini-Bressan (AoM, 2005): if initial data are of small BV, limit exists unique and is BV and is an entropy solution, (for linear viscosity).

Isothermal Dynamics

- J. Fritz, *Microscopic theory of isothermal elasticity*, ARMA 2011, infinite volume

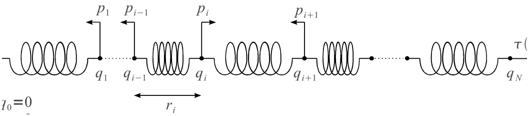
Isothermal Dynamics

- J. Fritz, *Microscopic theory of isothermal elasticity*, ARMA 2011, infinite volume
- S. Marchesani, S. Olla, Nonlinearity 2018, boundary conditions.

The system is in contact with a heat bath that keeps it at a constant temperature β^{-1}.
Energy is not conserved anymore. Macroscopically we have a p-system:

$$\begin{aligned} \partial_t r(t,y) &= \partial_y p(t,y) \\ \partial_t p(t,y) &= \partial_y \tau[\beta, r(t,y)] \end{aligned}$$

MIcroscopic isothermal dynamics



$$\begin{cases} dr_1 = Np_1 dt + N\sigma_N \left(V'(r_2) - V'(r_1)\right) dt - \sqrt{2\beta^{-1}N\sigma_N}\, d\widetilde{w}_1 \\ dr_i = N(p_i - p_{i-1})dt + N\sigma_N \left(V'(r_{i+1}) + V'(r_{i-1}) - 2V'(r_i)\right) dt + \sqrt{2\beta^{-1}N\sigma_N}(d\widetilde{w}_{i-1} - d\widetilde{w}_i) \\ dr_N = N(p_N - p_{N-1})dt + N\sigma_N \left(V'(r_{N-1}) - V'(r_N)\right) dt + \sqrt{2\beta^{-1}N\sigma}\, d\widetilde{w}_{N-1} \\ dp_1 = N(V'(r_2) - V'(r_1))dt + N\sigma_N (p_2 - p_1)\, dt - \sqrt{2\beta^{-1}N\sigma_N}\, dw_1 \\ dp_i = N(V'(r_{i+1}) - V'(r_i))dt + N\sigma_N (p_{i+1} + p_{i-1} - 2p_i)\, dt + \sqrt{2\beta^{-1}N\sigma_N}(dw_{i-1} - dw_i) \\ dp_N = N(\bar{\tau}(t) - V'(r_N))dt + N\sigma_N (p_{N-1} - p_N)\, dt + \sqrt{2\beta^{-1}N\sigma_N}\, dw_{N-1}, \end{cases},$$

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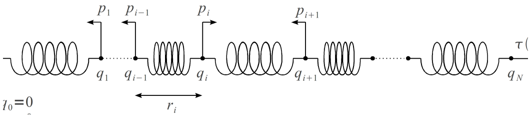

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\end{cases}
,
$$

$$
\lim_{N\to+\infty} \frac{\sigma_N}{N} = \lim_{N\to\infty} \frac{N}{{\sigma_N}^2} = 0.
$$

Isothermal dynamics, generator

$$\mathcal{G}_N^{\bar\tau(t)} := NL_N^{\bar\tau(t)} + N\sigma_N(S_N + \tilde S_N).$$

$$L_N^{\bar\tau(t)} = \sum_{i=1}^{N}(p_i - p_{i-1})\partial_{r_i} + \sum_{i=1}^{N-1}\left(V'(r_{i+1}) - V'(r_i)\right)\partial_{p_i} + (\bar\tau(t) - V'(r_N))\partial_{p_N},$$

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$$S_N := -\sum_{i=1}^{N-1} D_i^* D_i, \quad \tilde S_N := -\sum_{i=1}^{N-1} \tilde D_i^* \tilde D_i,$$

$$D_i := \frac{\partial}{\partial p_{i+1}} - \frac{\partial}{\partial p_i}, \qquad D_i^* := p_{i+1} - p_i - \beta^{-1} D_i$$

$$\tilde D_i := \frac{\partial}{\partial r_{i+1}} - \frac{\partial}{\partial r_i}, \qquad \tilde D_i^* := V'(r_{i+1}) - V'(r_i) - \beta^{-1}\tilde D_i.$$

Initial distribution

The density f_t^N with respect to $\mu^N = \mu^N_{\beta,0,0}$ solves the Fokker-Plank equation

$$\frac{\partial f_t^N}{\partial t} = \left(\mathcal{G}_N^{\bar\tau(t)}\right)^* f_t^N.$$

Here $\left(\mathcal{G}_N^{\bar\tau(t)}\right)^* = -NL_N^{\bar\tau(t)} + N\bar\tau(t)p_N + N\sigma(S_N + \tilde S_N)$ is the adjoint of $\mathcal{G}_N^{\bar\tau(t)}$ with respect to μ^N.

Initial distribution

The density f_t^N with respect to $\mu^N = \mu^N_{\beta,0,0}$ solves the Fokker-Plank equation

$$\frac{\partial f_t^N}{\partial t} = \left(\mathcal{G}_N^{\bar\tau(t)}\right)^* f_t^N.$$

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relative entropy

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assume or the initial distribution

$$H_N(f_0^N) \leq CN.$$

Limite Hydrodynamique

$$\frac{1}{N}\sum_x G(x/N)\begin{pmatrix} r_x(t) \\ p_x(t)\end{pmatrix} \underset{N\to\infty}{\longrightarrow} \int_0^1 G(y)\begin{pmatrix} r(y,t) \\ p(y,t)\end{pmatrix} dy$$

L^2-valued weak solution of

$$\begin{aligned} \partial_t r(t,y) &= \partial_y p(t,y) \\ \partial_t p(t,y) &= \partial_y \tau_\beta[r(t,y)] \\ p(t,0) = 0, &\quad \tau(r(t,1)) = \bar{\tau}(t), \end{aligned}$$

in the sense

$$\int_0^\infty \int_0^1 \left(r(t,x)\partial_t\varphi(t,x) - p(t,x)\partial_x\varphi(t,x)\right) dx\, dt = 0$$

$$\int_0^\infty \int_0^1 \left(p(t,x)\partial_t\psi(t,x) - \tau_\beta(r(t,x))\partial_x\psi(t,x)\right) dx\, dt = 0$$

for all functions φ, ψ with compact support on $\mathbb{R}_+ \setminus \{0\} \times (0,1)$.

NO information on initial and boundary conditions, no entropy condition.

Vanishing viscosity

The heat bath interaction in the dynamics plays the role of a *microscopic viscosity*, vanishing in the macroscopic limit.

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The corresponding viscous equations would be:

$$\begin{cases} \partial_t r^\varepsilon(t,x) - \partial_x p^\varepsilon(t,x) = \varepsilon \partial_{xx} \tau_\beta(r^\varepsilon(t,x)) & x \in (0,1) \\ \partial_t p^\varepsilon(t,x) - \partial_x \tau_\beta(r^\varepsilon(t,x)) = \varepsilon \partial_{xx} p^\varepsilon(t,x), & \end{cases}$$

with boundary conditions

$$p^\varepsilon(t,0) = 0, \quad \tau(r^\varepsilon(t,1)) = \bar\tau(t), \quad \partial_x p^\varepsilon(t,1) = 0, \quad \partial_x r^\varepsilon(t,0) = 0,$$

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Note the non-linear viscosity term.
As $\varepsilon \to 0$ boundary layers may appear.

The p-system

It is usually difficult to control bounds in the vanishing viscosity $\varepsilon \to 0$,
Bressan-Bianchini can do it for the BV if viscosity is taken linear.
For the p-system with no boundaries

$$\partial_t r(t,y) = \partial_y p(t,y)$$
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can be proven existence of L^∞ weak solutions (Di Perna), and L^2 valued solutions (Schearer, Serre-Shearer).
When boundaries are present, it is less clear how to define weak solutions that are not of BV.

Viscosity solutions with boundary values

One proposal would be to take L^2 limits as $\varepsilon \to 0$ of

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The non-linearity in the viscosity gives the right *entropy production*.

Entropy production and Clausius inequality

Let $v^\varepsilon(t,y) = r^\varepsilon(t,y), p^\varepsilon(t,y)$. Free energy at time t:

$$\mathcal{F}(v^\varepsilon(t)) = \int_0^1 \left[\frac{p^\varepsilon(t,y)^2}{2} + F_\beta(r^\varepsilon(t,y)) \right] dy, \qquad \partial_r F_\beta(r) = \tau_\beta(r),$$

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$$\begin{aligned}\mathcal{F}(v^\varepsilon(t)) - \mathcal{F}(v(0)) &= W(t) \\ &\quad - \varepsilon \int_0^t ds \int_0^1 dy \left[\left(\partial_y \tau_\beta(r^\varepsilon(s,y))\right)^2 + (\partial_x p^\varepsilon(s,y))^2\right] \\ &\geq W(t)\end{aligned}$$

where $W(t)$ is the work done by the boundary force $\tau(t)$.
So we expect that this particular limit generates the *right* entropy solutions.

Various remarks

- For scalar equations (one conserved quantity) the theory of boundary conditions is much simpler, and boundary layers do not depend on the detalis of the approximation (Bardos-Leroux-Nedelec, F. Otto), and there is a good definition of *boundary entropy condition*.

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- Hydrodynamic limit for one conserved quantity (Burgers equation) with boundary conditions have been proven by Bahadoran (from ASEP).
- There exists extention to systems of the *boundary entropy condition* (Chen-Frid), but with BV solutions.

Compensated compactness

- The Fritz's approach that we use is based on a stochastic version of the compensated compactness lemma of Tartar-Murat. This was used by Di Perna to prove existence of vanishing viscosity limits in p-systems.

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- This is a trick to prove that weak limit of viscous solutions v^{ve} are actually strong limit, which is also the main problem in hydrodynamic limits from microscopic dynamics.
- Unfortunately the trick works only when one has many (at least two) entropy pairs $((\eta_1, q_1), (\eta_2, q_2))$. This restrict the trick to 2x2 systems of conservation law, cannot be used for the Euler equation 3x3, where we know only the thermodynamic entropy as *mathematical entropy*.

Compensated compactness

$$\eta_j(v^\varepsilon)_t + q_j(v^\varepsilon)_x \in \text{compact set in} H^{-1}, \qquad j = 1, 2$$

then

$$\eta_1(v^\varepsilon)q_2(\varepsilon^\varepsilon) - \eta_2(v^\varepsilon)q_1(v^\varepsilon) \qquad \text{converge weakly in } L^\infty,$$

and this is enough to establish the strong convergence of v^ε.