LARGE DEVIATIONS FOR THE WIENER SAUSAGE

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§ OUTLINE

The focus in this lecture is on the Wiener sausage:

$$W_t = \bigcup_{s \in [0,t]} B_1(\beta_s), \qquad t \geq 0.$$

Here, $(\beta_s)_{s \geq 0}$ is Brownian motion on $\mathbb{R}^d$, and $B_1(x)$ is the closed ball of radius 1 centred at $x \in \mathbb{R}^d$.

Part 1:
Basic facts, LLN, CLT

Part 2:
LDP

PART 1

Basic facts, LLN, CLT

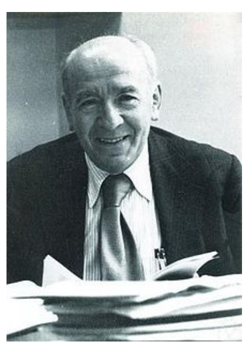
§ WIENER SAUSAGE

Let $\beta = (\beta_s)_{s \geq 0}$ be Brownian motion on $\mathbb{R}^d$, i.e., the continuous-time Markov process with generator Δ (the Laplace operator). Write $\mathbb{P}_x$ to denote the law of β starting from x, and put $\mathbb{P} = \mathbb{P}_0$.

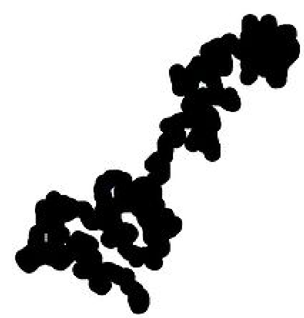
The Wiener sausage at time t is the random set

$$W_t = \bigcup_{s \in [0,t]} B_1(\beta_s), \qquad t \geq 0,$$

i.e., the Brownian motion drags around a ball of radius 1, which traces out a sausage-like environment.



Mark Kac 1974



The Wiener sausage is an important object because it is one of the simplest examples of a non-Markovian functional of Brownian motion. It plays a key role in the study of various stochastic phenomena:

- heat conduction
- trapping in random media
- spectral properties of random Schrödinger operators
- Bose-Einstein condensation

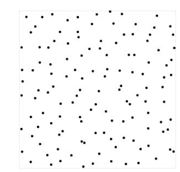
We will look at two specific quantities:

$$\mathcal{V}_t = \mathrm{Vol}(W_t),$$
$$\mathcal{C}_t = \mathrm{Cap}(W_t).$$

Here, Vol stands for volume and Cap for capacity. Only the case $d \geq 2$ is interesting.

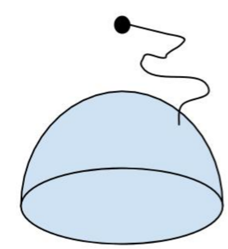
- The volume plays a role in trapping phenomena. Write PPP_α to denote the law of a Poisson Point Process on $\mathbb{R}^d$ with intensity $\alpha \in (0,\infty)$. Place balls of radius 1 around the points. Let τ be the first time that β hits a ball. Then

$$\mathsf{PPP}_\alpha(\tau > t) = e^{-\alpha \mathcal{V}_t}, \qquad t \geq 0.$$



- The capacity plays a role in hitting phenomena. Let $\bar{\tau}_{W_t}$ be the first time an auxiliary (!) Brownian motion $\bar{\beta}$ hits W_t. Then

$$\lim_{|x|\to\infty} |x|^{d-2}\,\bar{\mathbb{P}}_x\big(\bar{\tau}_{W_t} < \infty\big) = \frac{\mathcal{C}_t}{\kappa_d}, \qquad t \geq 0, \quad d \geq 3,$$



with $\kappa_d = \mathrm{Cap}(B_1(0))$.

Another interpretation is that $\kappa_d/\mathcal{C}_t$ equals the minimal electrostatic energy of a unit charge distributed on W_t.

Other set functions are of interest too:

> perimeter, moment of inertia, principal Dirichlet eigenvalue, heat content, torsional rigidity.

Many have the property that they are either minimal or maximal when the set is a ball, which is analytically helpful.

In what follows we recall some basic facts, and state LLNs and CLTs. After that we focus on LDPs. It will turn out that there is a remarkable dependence on the dimension d.

NOTE: By Brownian scaling, it is trivial to transfer all the properties to be described below to the Wiener sausage with radius $r \in (0, \infty)$ rather than $r = 1$.

§ VOLUME

The following strong LLN holds:

$$d \geq 3: \quad \lim_{t\to\infty} \frac{1}{t} \mathcal{V}_t = \kappa_d \quad \mathbb{P} - a.s.$$

Spitzer 1964

where

$$\kappa_d = \mathrm{Cap}(B_1(0)) = \frac{4\pi^{d/2}}{\Gamma(\frac{d-2}{2})}.$$

The existence of the limit follows from the subaddivity property

$$\mathrm{Vol}(A \cup B) \leq \mathrm{Vol}(A) + \mathrm{Vol}(B) \qquad \forall A, B \subset \mathbb{R}^d \text{ Borel.}$$

The identification of the limit requires potential theory.

The case $d = 2$ is critical:

$$\lim_{t\to\infty} \frac{\log t}{t} \mathcal{V}_t = \kappa_2 = 4\pi \quad \mathbb{P} - a.s.$$

Le Gall 1988

The CLT holds with

$$\mathbb{V}\mathrm{ar}(\mathcal{V}_t) \asymp \begin{cases} t^2/\log^4 t, & d = 2, \\ t \log t, & d = 3, \\ t, & d \geq 4, \end{cases}$$

Spitzer 1964, Le Gall 1988

The limit law is Gaussian for $d \geq 3$ and non-Gaussian for $d = 2$.

§ CAPACITY

The (Newtonian) capacity of a Borel set $A \subset \mathbb{R}^d$ can be defined through the variational formula

$$\frac{1}{\mathrm{Cap}(A)} = \inf_{\mu \in \mathcal{P}(A)} \int_A \mu(dx) \int_A \mu(dy)\, G_d(x,y),$$

where $\mathcal{P}(A)$ is the set of probability measures on A, and

$$G_d(x,y) = \frac{1}{\kappa_d |x-y|^{d-2}}, \qquad x,y \in \mathbb{R}^d,$$

is the Green function. Think of the above integral as the electrostatic energy when the set A is a conductor with a unit charge.

A different formula is needed for the case $d = 2$, which we will not consider.

The following strong LLN holds:

$$d \geq 5: \quad \lim_{t\to\infty} \frac{1}{t} \mathcal{C}_t = c_d \quad \mathbb{P} - a.s.$$

Asselah, Schapira, Sousi 2017

The existence of the limit follows from the subaddivity property

$$\text{Cap}(A \cup B) \leq \text{Cap}(A) + \text{Cap}(B) \qquad \forall A, B \subset \mathbb{R}^d \text{ Borel.}$$

The fact that the limit is non-degenerate requires some estimates. No explicit formula is known for c_d.

The case $d = 4$ is critical:

$$\lim_{t\to\infty} \frac{\log t}{t}\mathcal{C}_t = c_4 = 8\pi^2 \quad \mathbb{P}-a.s.$$

Asselah, Schapira, Sousi 2016

The CLT is expected (!) to hold with

$$\mathbb{V}\mathrm{ar}(\mathcal{C}_t) \asymp \begin{cases} t^2/\log^4 t, & d = 4, \\ t\log t, & d = 5, \\ t, & d \geq 6, \end{cases}$$

Asselah, Schapira, Sousi 2016 – discrete setting

The limit law is expected (!) to be Gaussian for $d \geq 5$ and non-Gaussian for $d = 4$.

The cases $d = 2, 3$ are interesting too, but are less well understood. We will not consider them.

PART 2

LDP

§ LARGE DEVIATIONS FOR THE VOLUME

THEOREM 1 Van den Berg, Bolthausen, dH, 2001 + 2004

Let $d \geq 3$*. For every* $b > 0$*,*

$$\lim_{t\to\infty} \frac{1}{t^{(d-2)/d}} \log \mathbb{P}(\mathcal{V}_t \leq bt) = -I_d^{\downarrow}(b),$$

where

$$I_d^{\downarrow}(b) = \inf_{\phi\in\Phi_d(b)} \int_{\mathbb{R}^d} |\nabla\phi|^2(x)\,dx$$

with

$$\Phi_d(b) = \Big\{\phi \in H^1(\mathbb{R}^d)\colon\ \int_{\mathbb{R}^d} \phi^2(x)\,dx = 1,\ \int_{\mathbb{R}^d} \Big(1 - e^{-\kappa_d \phi^2(x)}\Big)\,dx \leq b\Big\}.$$

LDP

HEURISTICS:

The idea is that the optimal strategy for β to realise the event $\{\mathcal{V}_t \leq bt\}$ is to behave like a Brownian motion in a drift field

$$t^{1/d}x \mapsto (\nabla\phi/\phi)(x) \qquad \text{for some } \phi \in \Phi_d(b).$$

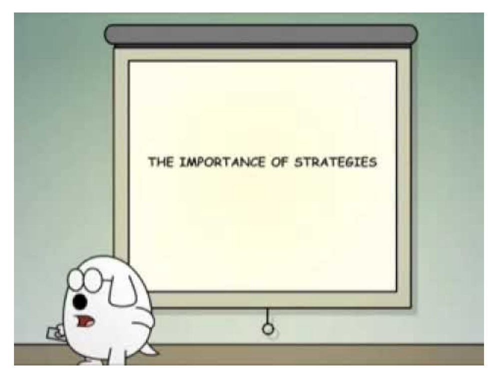
The cost of adopting this drift during a time t is

$$\exp[t^{(d-2)/d}\,\|\nabla\phi\|_2^2],$$

to leading order. Conditional on adopting this drift, the path spends time $t\phi^2(x)dx$ in the volume element $t^{1/d}dx$ and the Wiener sausage covers a fraction

$$1 - e^{-\kappa_d \phi^2(x)}$$

of this volume element.



CONCLUSION:

The optimal strategy for the Wiener Sausage to achieve a downward large deviation of its volume is to look like a Swiss cheese!

THEOREM 2 Van den Berg, Bolthausen, dH, 2001 + 2004

Let $d = 2$. *For every* $b > 0$,

$$\lim_{t\to\infty} \frac{1}{\log t} \log \mathbb{P}(\mathcal{V}_t \le bt/\log t) = -I_2^{\downarrow}(b),$$

where

$$I_2^{\downarrow}(b) = \inf_{\phi\in\Phi_2(b)} \int_{\mathbb{R}^2} |\nabla\phi|^2(x)\,dx$$

with

$$\Phi_2(b) = \left\{\phi \in H^1(\mathbb{R}^2) \colon \int_{\mathbb{R}^2} \phi(x)^2\,dx = 1, \int_{\mathbb{R}^2} \left(1 - e^{-\kappa_2\phi(x)^2}\right) dx \le b\right\}.$$

LDP

NOTE:

The cost of a downward large deviation in the critical dimension $d = 2$ is polynomial rather than stretched exponential.

§ RATE FUNCTIONS

By Brownian scaling, the variational formula for the rate function can be standardised, namely,

$$I_d^{\downarrow}(b) = \kappa_d^{-2/d} \chi_d\left(\frac{b}{\kappa_d}\right), \qquad b > 0,$$

where

$$\chi_d(u) = \inf\Big\{\|\nabla\psi\|_2^2 \colon \psi \in H^1(\mathbb{R}^d), \|\psi\|_2 = 1, \int_{\mathbb{R}^d} (1 - e^{-\psi^2}) \leq u\Big\}.$$

The function χ_d is continuous on $(0,\infty)$, strictly decreasing on $(0,1)$, and equal to zero on $[1,\infty)$. If a minimiser exists, then it is unique modulo translations, radially symmetric and radially strictly decreasing.

Moreover,

$$\lim_{u \downarrow 0} u^{2/d} \chi_d(u) = -\lambda_d,$$

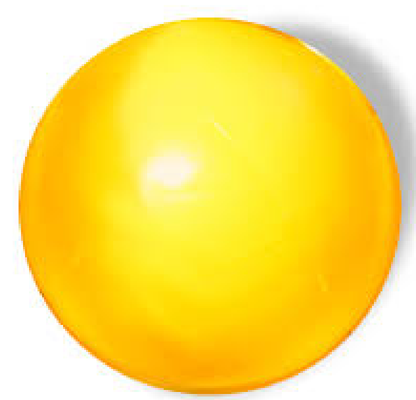
with $\lambda_d \in (0, \infty)$ the principal Dirichlet eigenvalue of $-\Delta$ on $B_1(0)$.

The latter corresponds to the regime where $\mathcal{V}_t = o(\mathbb{E}(\mathcal{V}_t))$ and the Swiss cheese squeezes out its holes to becomes a ball. This regime was studied earlier and is much easier to handle.

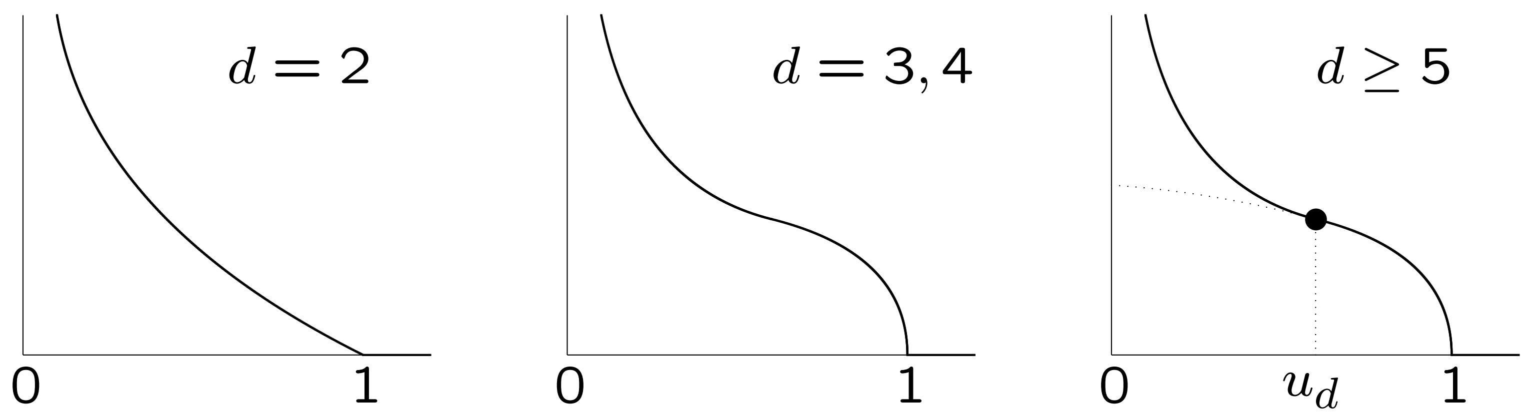
Donsker, Varadhan 1975

Bolthausen 1990 + 1994

Sznitman 1990



Qualitative picture of $u \mapsto \chi_d(u)$:



For $d \geq 5$, there is a critical value $u_d \in (0, 1)$ above which the variational formula has no minimiser: mass leaks a way to infinity.

The optimal strategy is time-inhomogeneous: partly on scale $t^{1/d}$, partly on scale $\sqrt{t}$.

§ WHAT ABOUT UPWARD LARGE DEVIATIONS?

Upward large deviations are more costly. For $d \geq 3$,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}(\mathcal{V}_t \geq bt) = -I_d^{\uparrow}(b), \qquad b > 0,$$

Hamana, Kesten 2001

with

$$I_d^{\uparrow}(b) > 0 \qquad \forall\, b > \kappa_d.$$

Upward large deviations are much harder to capture than downward large deviations: they require a local dilation of the Wiener sausage essentially everywhere.

No variational formula is known for $I_d^{\uparrow}$. Only bounds are available via estimates on exponential moments.

van den Berg, Tóth 1991

van den Berg, Bolthausen 1994

§ LARGE DEVIATIONS FOR THE CAPACITY

THEOREM 3 Van den Berg, Bolthausen, dH, in progress

Let $d \geq 5$. *For every* $b > 0$,

$$\lim_{t\to\infty} \frac{1}{t^{(d-4)/(d-2)}} \log \mathbb{P}(\mathcal{C}_t \leq bt) = -I_d^{\downarrow}(b),$$

where

$$I_d^{\downarrow}(b) = \inf_{\phi\in\Phi_d(b)} \int_{\mathbb{R}^d} |\nabla\phi|^2(x)\,dx$$

with

$$\Phi_d(b) = \left\{\phi \in H^1(\mathbb{R}^d)\colon\ \int_{\mathbb{R}^d} \phi^2(x)\,dx = 1,\ \int_{\mathbb{R}^d} \phi^2(x)\,[c_d u_\phi(x)]\,dx \leq b\right\},$$

LDP

where u_ϕ is the unique solution of the equation

$$\Delta u = \left(\frac{c_d}{2d}\,\phi^2\right) u, \qquad u(\infty) \equiv 1,$$

Note that the function u_ϕ is the solution of the Schrödinger equation with potential $-\frac{c_d}{2d}\,\phi^2$ and with boundary condition 1 at infinity.

HEURISTICS:

The idea is that the optimal strategy for β to realise the event $\{\mathcal{C}_t \leq bt\}$ is to behave like a Brownian motion in a drift field

$$t^{1/(d-2)}x \mapsto (\nabla\phi/\phi)(x) \qquad \text{for some } \phi \in \Phi_d(b).$$

The cost of adopting this drift during a time t is

$$\exp[t^{(d-4)/(d-2)}\,\|\nabla\phi\|_2^2],$$

to leading order. Conditional on adopting this drift, the path spends time $t\phi^2(x)dx$ in the volume element $t^{1/(d-2)}dx$ and the capacity associated with the Wiener sausage in this volume element is

$$u_\phi(x)c_d\,[t\phi^2(x)dx].$$

In the latter expression:

- c_d is the probability that $\bar{\beta}$ escapes locally from W_t, i.e., moves out of the volume element $t^{1/(d-2)}dx$ without hitting W_t.

- $u_\phi(x)$ is the probability that β escapes globally from W_t, i.e., moves to infinity without hitting the part of W_t that lies outside the volume element $t^{1/(d-2)}dx$.

For capacity both local and global properties control the downward large deviations.

CONCLUSION:

The optimal strategy for the Wiener Sausage to achieve a downward large deviation of its capacity is to look like thin Italian spaghetti!

THEOREM 4 Van den Berg, Bolthausen, dH, in progress

Let $d = 4$. *For every* $b > 0$,

$$\lim_{t \to \infty} \frac{1}{\log t} \log \mathbb{P}(\mathcal{C}_t \leq bt/\log t) = -I_4^{\downarrow}(b),$$

where

$$I_4^{\downarrow}(b) = \inf_{\phi \in \Phi_4(b)} \int_{\mathbb{R}^4} |\nabla \phi|^2(x)\, dx$$

with

$$\Phi_4(b) = \Big\{ \phi \in H^1(\mathbb{R}^4) \colon \int_{\mathbb{R}^4} \phi(x)^2\, dx = 1, \quad \int_{\mathbb{R}^4} \phi(x)^2 \, [c_4 u_\phi(x)]\, dx \leq b \Big\},$$

where u_ϕ *is the unique solution of the equation*

LDP

$$\Delta u = \left(\frac{c_4}{8}\phi^2\right) u, \qquad u(\infty) \equiv 1.$$

The cost of a downward large deviation in the critical dimension $d = 4$ is polynomial rather than stretched exponential.

§ RATE FUNCTIONS

By Brownian scaling, we get

$$I_d^{\downarrow}(b) = \left(\frac{c_d}{2d}\right)^{-2/(d-2)} \chi_d\left(\frac{b}{c_d}\right), \qquad b > 0,$$

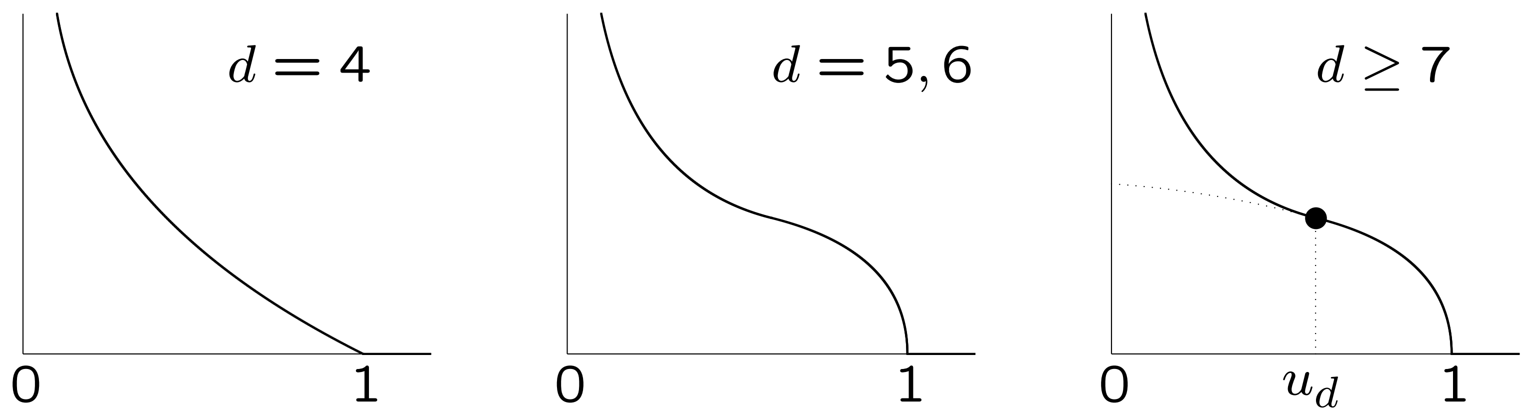
where

$$\chi_d(u) = \inf\Big\{\|\nabla\psi\|_2^2\colon\ \psi \in H^1(\mathbb{R}^d),\ \|\psi\|_2 = 1,\ \textstyle\int_{\mathbb{R}^d} \psi^2 u_\psi \le u\Big\}$$

with u_ψ the solution of $\Delta u = \psi^2 u$, $u(\infty) \equiv 1$.

We expect that upward large deviations are exponentially costly, but no results are yet available.

Qualitative picture of $u \mapsto \chi_d(u)$:



REMARKABLE: The scaled rate function for the capacity in dimension $d+2$ is qualitatively the same as that for the volume in dimension d.